# Population Protocols over Ordered Agents

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[4] MPI for Software Systems, Saarbrücken, Germany

# Population protocols

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Distributed computing consisting of an unbounded number of passively mobile, finite-state agents

**Angluin & al. (PODC'04)**

- Anonymous
- two-by-two interactions

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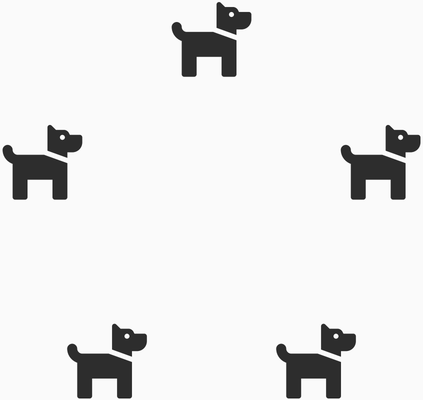

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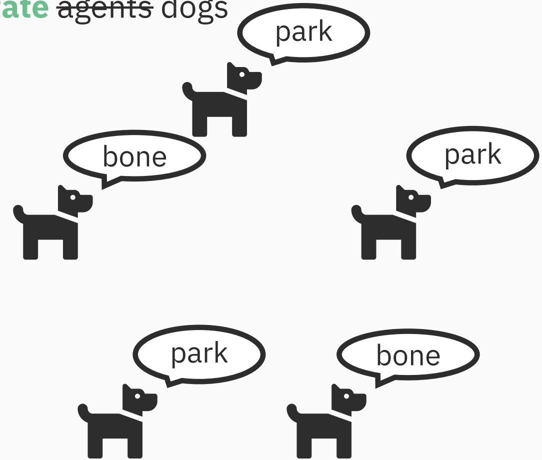

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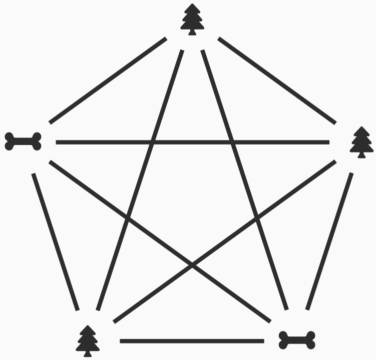

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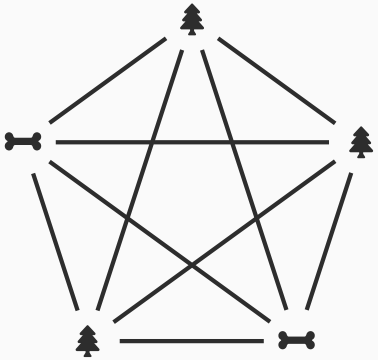

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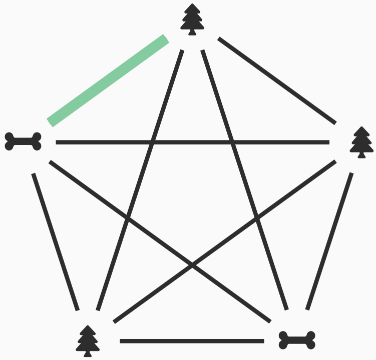

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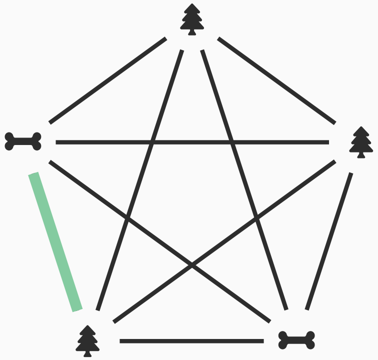

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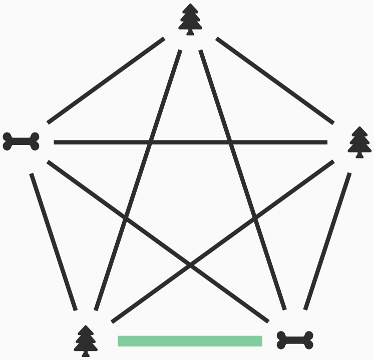

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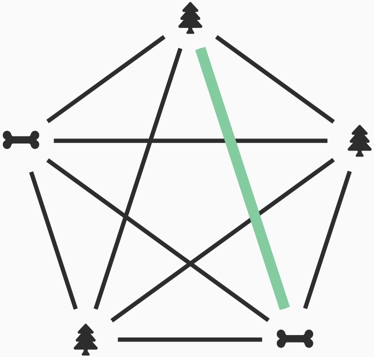

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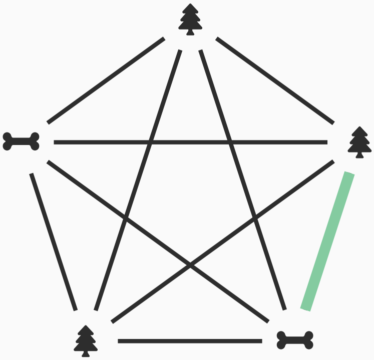

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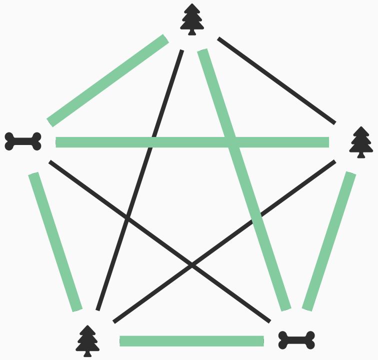

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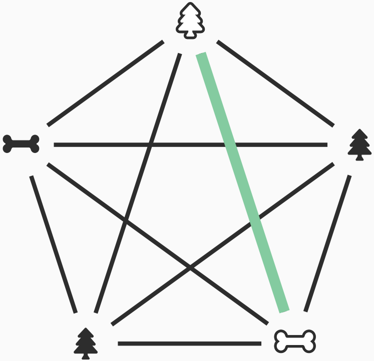

# Example: Majority

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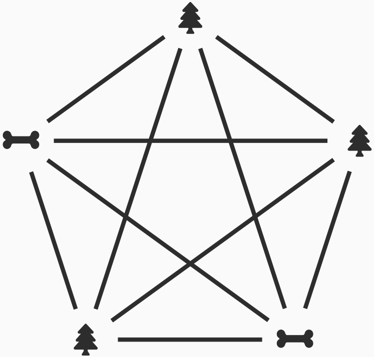
This protocol computes whether initially there are more  or 



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- Opposite active agents become passive
- Active agent convince passive agent

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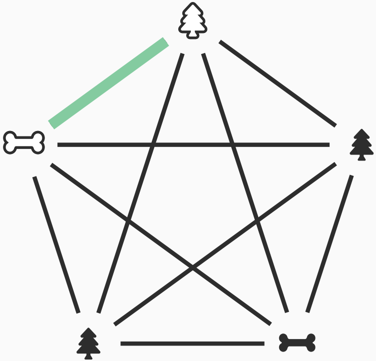

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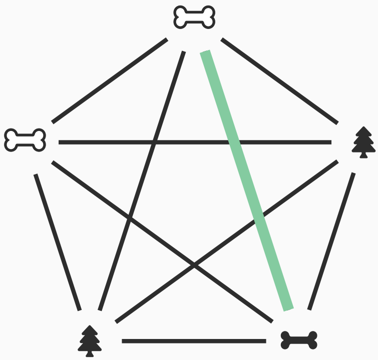

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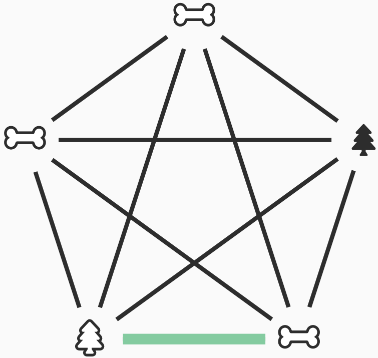

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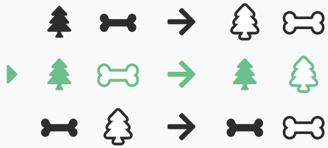

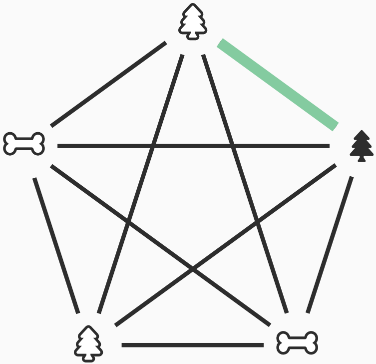

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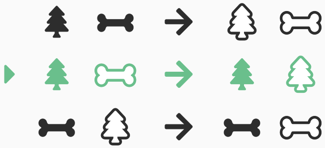

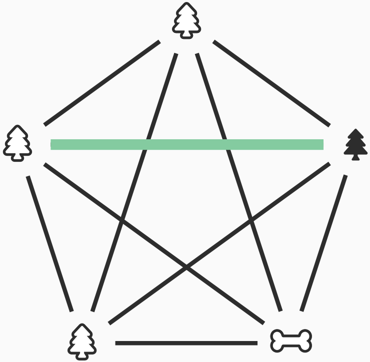

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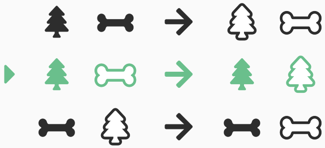

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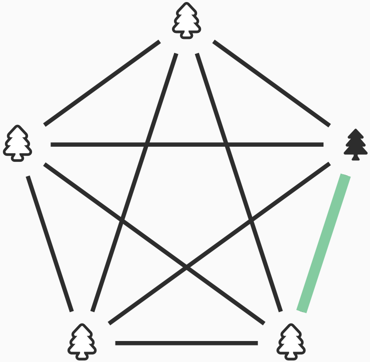

# Population protocols: formal model

| | |
|---|---|
| States: | finite set $Q$ |
| Input mapping: | $\iota\colon \Sigma \to Q$ with finite $\Sigma$ |
| Output mapping: | $O\colon Q \to \{0,1\}$ |
| Transitions: | $\to \subseteq Q^2 \times Q^2$ |

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$$
\begin{aligned}
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**Theorem** — Angluin & al. (PODC'06)

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- Protocol computes a predicate $\varphi\colon \mathbb{N}^\Sigma \to \{0, 1\}$ if, with probability 1, the agents eventually reach a unanimous and permanent agreement.

# Population protocols: formal model

A protocol is immediate-observation (IO-PP) if a single agent can change in a interaction, i.e. transitions are of this form :

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**Majority transitions** :

non-immediate-observation

immediate-observation

Is not immediate-observation because the first transition changes both

# Population protocols over ordered agents (PP[<])

**What happens if agents have unique identifiers ?**

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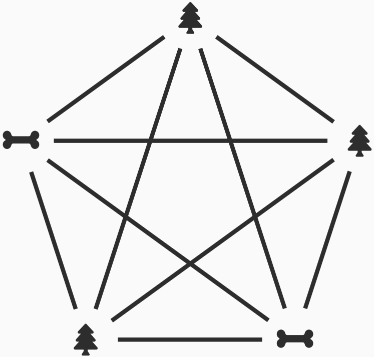

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- ~~Anonymous~~ **With ordered IDs**
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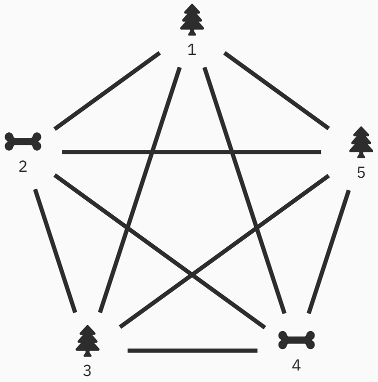

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- ~~Anonymous~~ With ordered IDs
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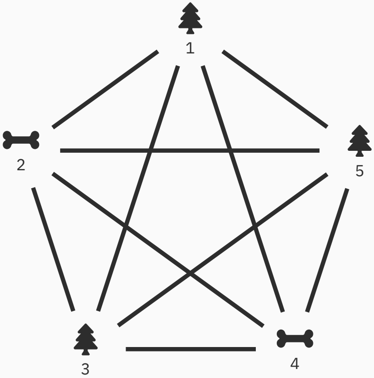

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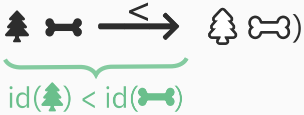
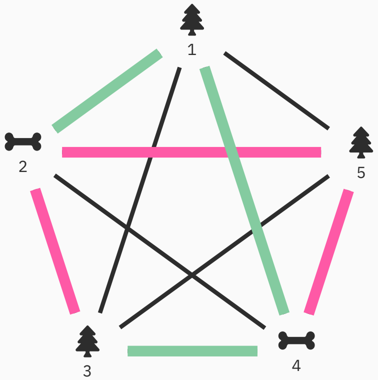
  (e.g. )


**: Possible** **: Impossible**

# Example: the growling language

# Example: a growling protocol

This protocol decides wether $w \in GR^*$

- $Q = \{\vdash^{\text{👍}}_{G}, -^{\text{👍}}_{R}, \vdash^{\text{👎}}_{G}, -^{\text{👎}}_{G}, -^{\text{👎}}_{R}\}$
- $\iota(G) = \vdash^{\text{👍}}_{G}, \quad \iota(R) = -^{\text{👎}}_{R}$
- $O(*^{\text{👍}}_{*}) = 1, \quad O(*^{\text{👎}}_{*}) = 0$
- Transitions :

$$\vdash^{\text{👍}}_{G} \; -^{\text{👎}}_{R} \xrightarrow{<} \vdash^{\text{👍}}_{G} \; -^{\text{👍}}_{R} \;\Big|\; *^{*}_{*} \; \vdash^{\text{👍}}_{G} \xrightarrow{<} *^{*}_{*} \; -^{\text{👎}}_{G} \;\Big|\; \begin{array}{l} *^{*}_{*} \; -^{\text{👎}}_{G} \xrightarrow{<} *^{\text{👎}}_{*} \; -^{\text{👎}}_{G} \\ -^{\text{👎}}_{G} \; *^{*}_{*} \xrightarrow{<} -^{\text{👎}}_{G} \; *^{\text{👎}}_{*} \end{array}$$

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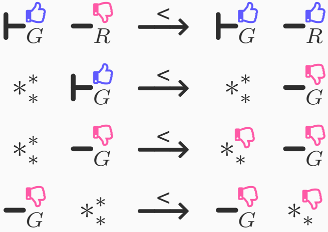
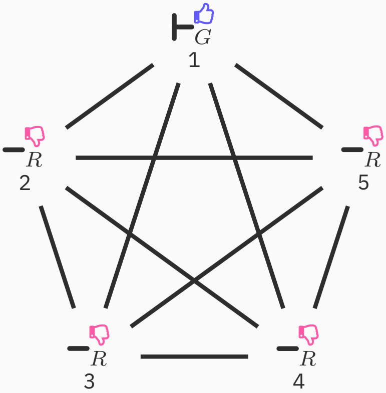


$\iota(GRRRR)$



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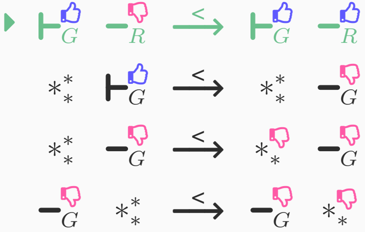


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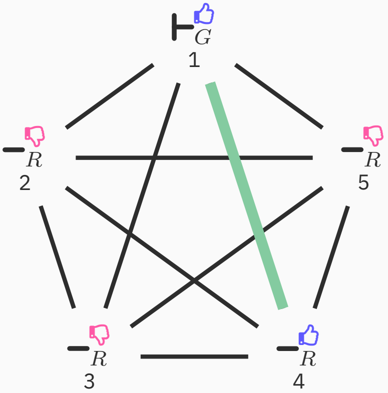

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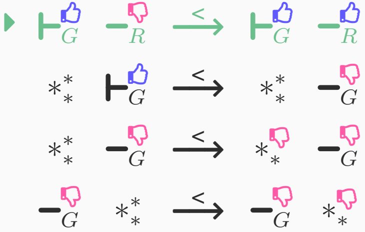

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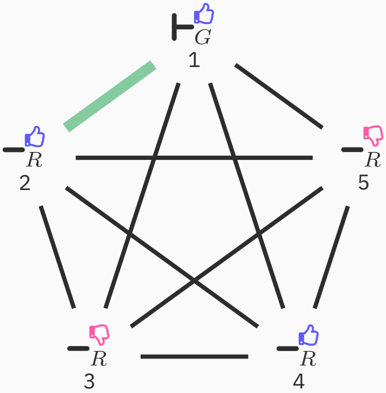

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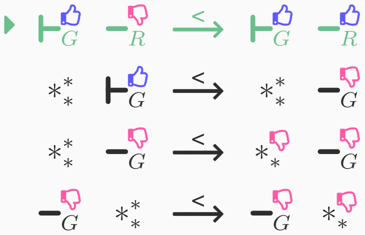


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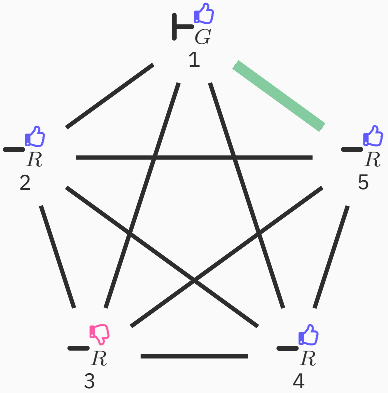

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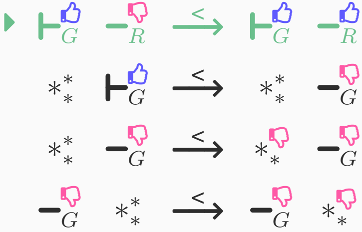


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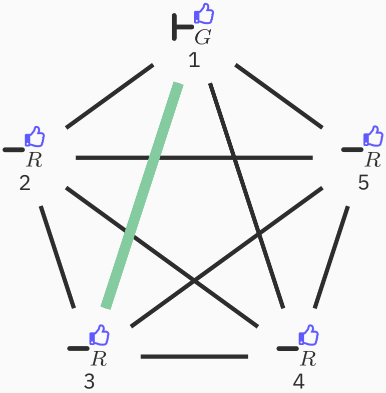

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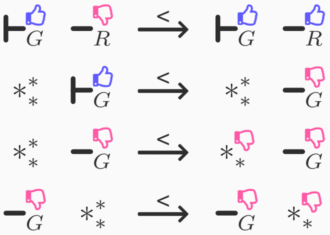

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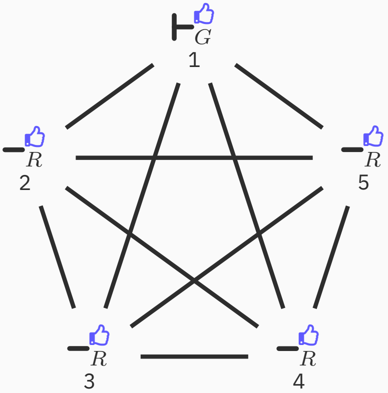

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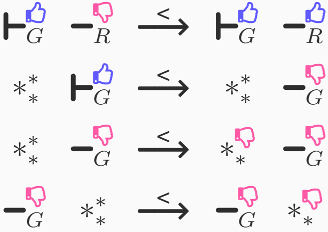


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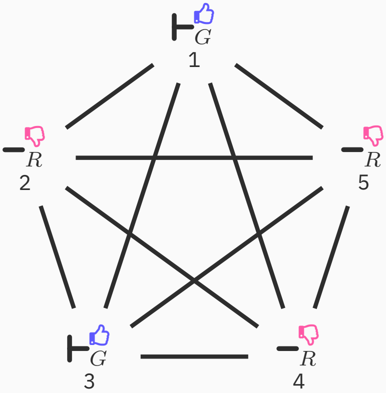

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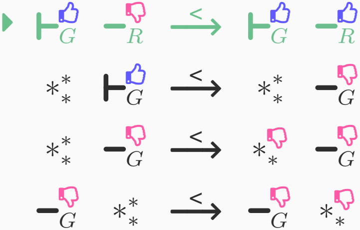

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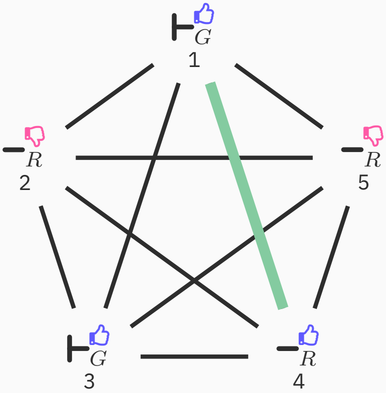

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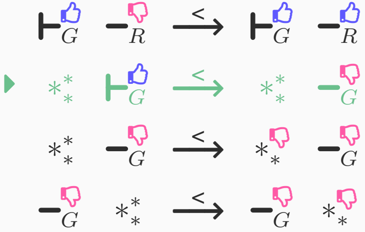

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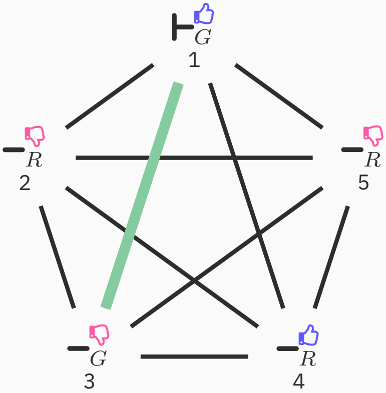

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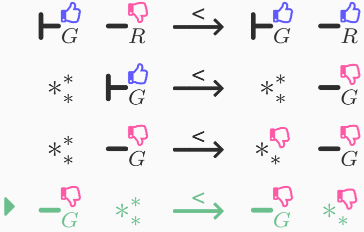

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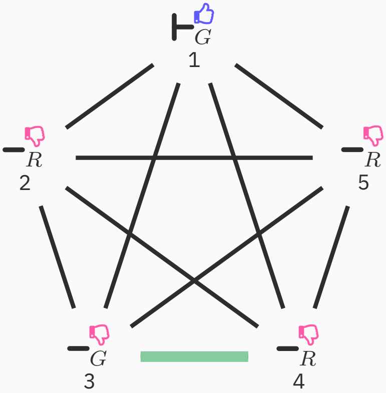

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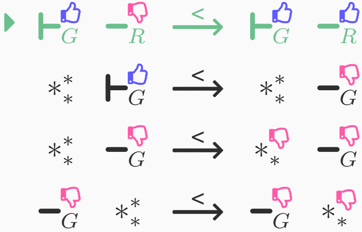


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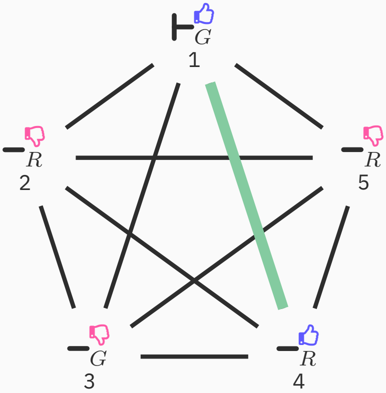
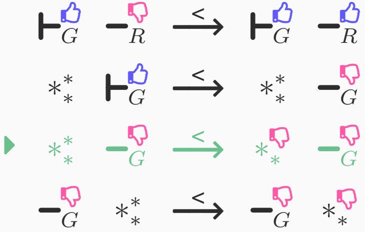


# Example: a growling protocol

This protocol decides wether $w \in GR^*$

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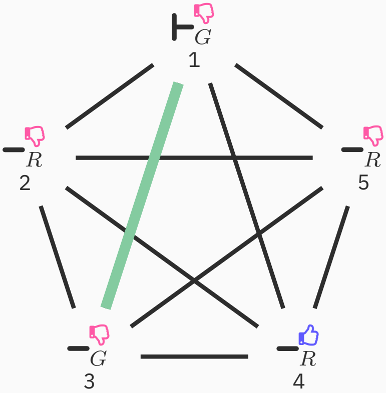

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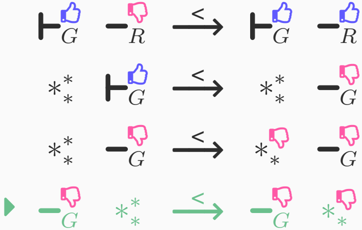


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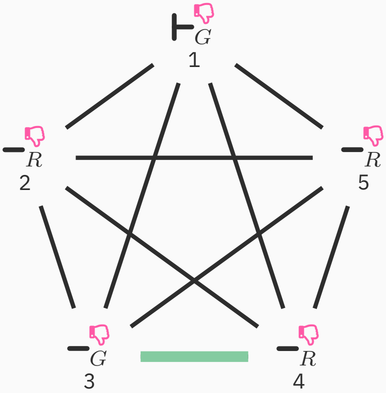

# Characterization of IO-PP[<]

# DA: unambiguous star-free languages

**Automata**

- partially ordered unambiguous automata
- partially ordered two-way deterministic automata

**DA**

**Algebra**

- languages recognized by monoids of the variety DA

**Logic**

- $\mathrm{LTL}[\mathbf{F}^{-1}, \mathbf{F}]$
- $\exists\forall\mathrm{FO}[<] \cap \forall\exists\mathrm{FO}[<]$
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**Distributed systems**

- IO-PP$[<]$

# Semi-decidability

- A protocol **decides** $L$ if, w.p. 1, the agents reach a permanent unanimous agreement

  $\boldsymbol{u} \in \boldsymbol{L} \;\Rightarrow\;$ eventually, all agents *accept* forever

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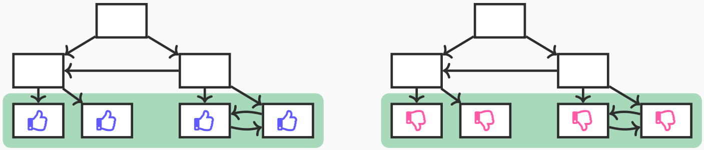

*Agreement 1 with probability 1*

*Agreement 0 with probability 1*

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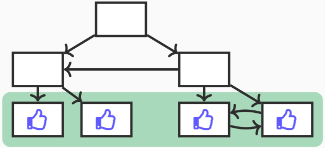

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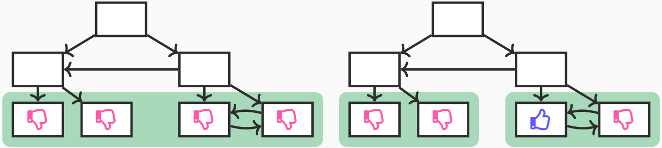

*Agreement 0 with probability 1* *Agreement 0* *No stable agreement*

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**Lemma** **Our work (ICALP'26)**

If protocol and its complement are semi-deciders for $\boldsymbol{L}$ and $\Sigma^+ \setminus \boldsymbol{L}$, then $\boldsymbol{L}$ recognized by a protocol.

# IO-PP[<] $\supseteq$ DA

- $DA = \exists\forall\,\mathrm{FO}[<] \cap \forall\exists\,\mathrm{FO}[<]$ **Pin & Weil (ICALP'95)**

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guess role

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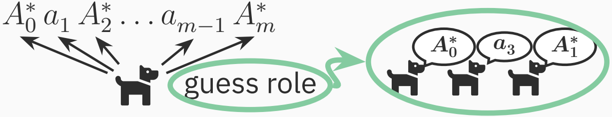

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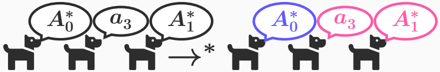

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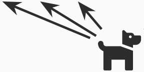


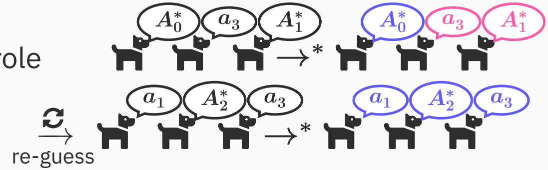

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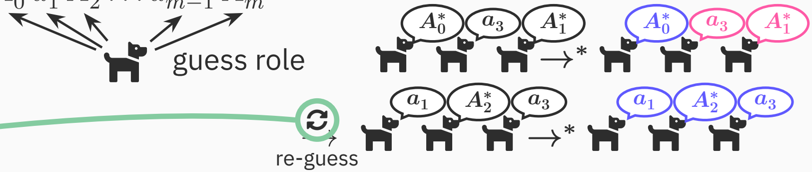


- **stabilizing input**: agents can change their mind about their input.

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**Pin & Weil (ICALP'95)**

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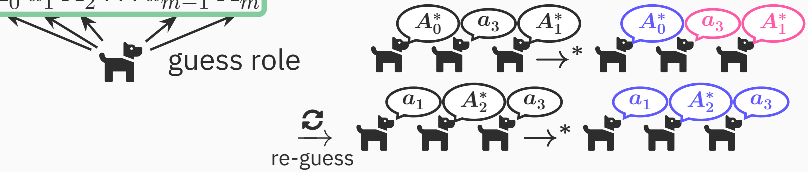


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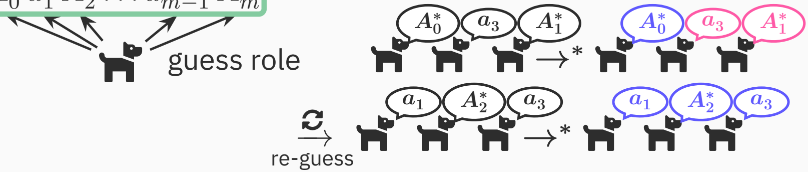


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**Lemma** **Our work (ICALP'26)**

Semi-deciders are closed under ∪ and ∩

# IO-PP[<] $\subseteq$ DA

**Language $L$ is in DA iff :**

- For all $\boldsymbol{W} \equiv_{\boldsymbol{L}} \boldsymbol{W}\boldsymbol{W}, \quad \boldsymbol{W} \equiv_{\boldsymbol{L}} \boldsymbol{W}\ h\boldsymbol{W}$ with $\boldsymbol{\alpha}(\boldsymbol{h}) \subseteq \boldsymbol{\alpha}(\boldsymbol{W})$

- $L$ is regular

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$$w_1 \equiv_L w_2 \Leftrightarrow \forall x, y \cdot (x\, w_1\, y \in L \Leftrightarrow x\, w_2\, y \in L)$$

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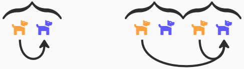

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only the observer changes, not the observed

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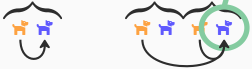

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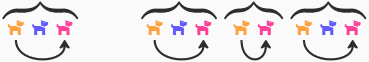

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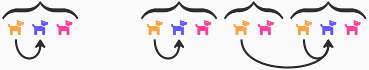

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- $L$ is regular
  - Every word long enough contains a pattern that can be pumped to produce a new word in the same equivalence class.
  - Finitely many equivalence classes $\Rightarrow L \in$ IO-PP[<] is regular.

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**Algebra**

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**Distributed systems**

- IO-PP$[<]$

Our work (ICALP’26)

# Beyond IO-PP[<]

# PP[<] $\supseteq$ p.o. Parikh automata

**Theorem** — **Our work (ICALP'26)**

If $L$ and $\overline{L}$ are both recognized by p.o. Parikh automata, then $L$ is decided by a PP[<].

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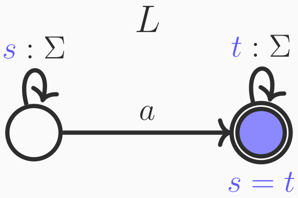
**E.g. Median language** : $L = \bigcup_{n \in \mathbb{N}} \Sigma^n a \Sigma^n$



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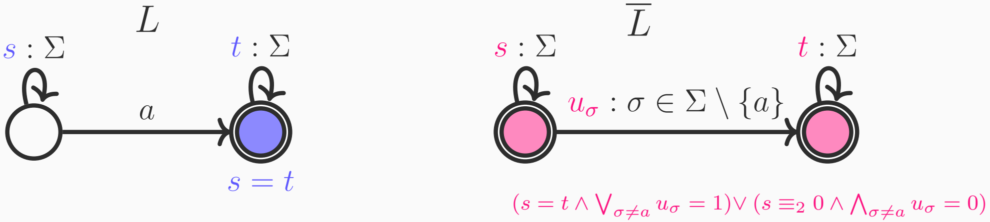
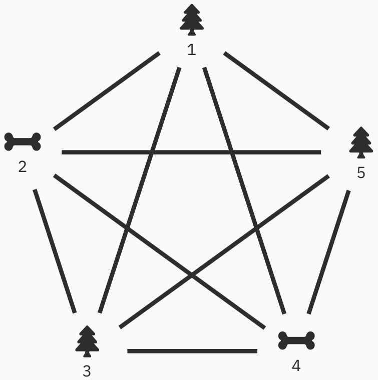


# PP[+1] = NSPACE(n)

Distributed computing consisting of an unbounded number of passively mobile, finite-state ~~agents~~ dogs

- ~~Anonymous~~ With ordered IDs
- Ordered two-by-two interactions (e.g. $\xrightarrow{+1}$ )



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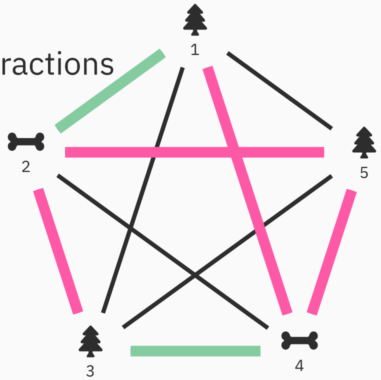
- ~~Anonymous~~ With ordered IDs
- **Neighbor** Ordered two-by-two interactions (e.g. 🌲 🦴 $\xrightarrow{+1}$ 🌲 🦴)

  id(🦴) = id(🌲) + 1




●: **Possible** ●: **Impossible**

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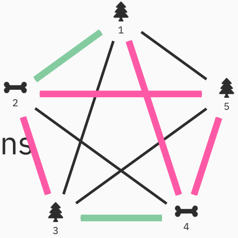


**Theorem** **Our work (ICALP'26)**

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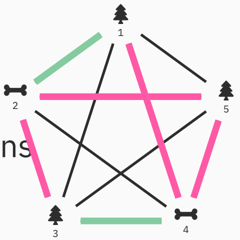


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PP[+1] $=$ IO-PP[+1] $=$ NSPACE(n)

# Conclusion

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## Summary

- Introduction to population protocols over ordered agents
- $\text{IO-PP}[<] = \text{DA}$
- $\text{IO-PP}[+1] = \text{PP}[+1] = \text{NSPACE}(n)$

## Future work

- Characterization of PP[<]

  Expressiveness ?

  State complexity ?

  Expected time complexity ?

# Conclusion

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Are unambiguous p.o. Parikh automata closed under complementation ?

# Thank you !